ON FINITE INTEGRAL TRANSFORMS BUILT UPON INCOMPLETE SETS OF EIGENFUNCTIONS

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Introduction. In the literature, solutions of boundary-value problems are often expressed in terms of incomplete sets of eigenfunctions. There does not seem to be any procedure that can be followed in general. And, the incomplete set discovered in the process of solving a particular problem is not systematically investigated to see if, or to what extent, it can be used in other problems. In this note, we will show that the method of integral transforms can be employed as a format in employing these incomplete sets to the greatest advantage.

We will demonstrate this with two examples. The first example is the finite Fourier sine transform with a part of the complete set of eigenfunctions omitted intentionally. The second example is a new transform (called an equitriangular transform) established by the author on an incomplete set of eigenfunctions in a equilateral triangular region. The first example is of course trivial. But it serves to show the motivation behind the procedure used. It will be seen in both examples that an integral transform based on an incomplete set of eigenfunction can only be used in solving a limited class of boundary-value problems. The limitations (beside the ones common to all integral transforms[1]) will show up in the admissible boundary values, forms of the inhomogeneous terms and initial values, if any. But these limitations will come out of the procedure, and are known beforehand. The

[1] Linearity, appearance of only powers of the Laplacian, particular kind of boundary conditions, etc.

limited class of boundary-value problems will still be general enough to embody many physical problems.

The procedure is mainly aimed at problems in two-dimensional regions which can not be decomposed into two one-dimensional regions; i.e., regions other than rectangles, circles, annulus and sectors. For such regions, there is no systematical way[2] of obtaining a complete set of eigenfunctions; but an incomplete set is often known in the literature. It is then possible to find the class of problems whose solutions can be expressed in terms of each one of these incomplete sets.

A Simple Example.

The complete set of eigenfunctions of the system

$$\begin{cases} f''(x) + \lambda^2 f(x) = 0, & 0 < x < X \\ f = 0 \quad \text{at} \quad x = 0 & \text{and } X \end{cases}$$

is known to be $\sin \frac{m\pi x}{X}$, $m = 1, 2, 3, 4, \ldots$

Based on this complete set, we can establish the finite Fourier sine transform[3] for functions satisfying Dirichlet's conditions in (0, X)

$$\bar{F}(m) = \int_0^X F(x) \sin \frac{m\pi x}{X} \, dx$$

which has the important properties that

$$\overline{F''} = \frac{m\pi}{X} \left[(-1)^{m+1} F(X) - F(0) \right] - \frac{m^2 \pi^2}{X^2} \bar{F}$$

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[2]Except that of an approximate nature based on the variational principle.

[3]Sneddon, I. N., Fourier Transforms, McGraw-Hill, 1951, pp. 71-76.

and

$$\frac{1}{2}\left[F(x+0) + F(x-0)\right] = \frac{2}{X} \sum_{m=1}^{\infty} \bar{F} \sin \frac{m\pi x}{X}$$

This transform can then be employed to solve linear boundary value problems that contain only even-ordered derivatives, with functional values given on $x = 0$ and X.

Now, for the sake of argument, let us assume that we are only able to obtain the incomplete set of eigenfunctions

$$\sin \frac{n\pi x}{X} , \quad n = 1, 3, 5, 7 \ldots$$

We can also establish a transform

$$\tilde{F}(n) = \int_0^X F \sin \frac{n\pi x}{X} dx$$

But then,

$$\widetilde{F''} = -\frac{n\pi}{X}\left[F(X) - F(0)\right] - \frac{n^2\pi^2}{X^2} \tilde{F}$$

and the transform is no longer of immediate use in boundary-value problems. For one thing, the transform of F'' recognizes only the difference between $F(X)$ and $F(0)$, not $F(X)$ and $F(0)$ individually.

We can remedy this, of course, by supplementing this incomplete set of eigenfunctions by the supposedly unknown complementary set $\left\{\phi_\nu(x, \xi_\nu)\right\}$ associated with eigen-values ξ_ν (also supposedly unknown). Then, for functions satisfying Dirichlet's conditions, we have

$$\frac{1}{2}\left[F(x+0) + F(x-0)\right] = \frac{2}{X} \sum_{n=1,3,5}^{\infty} \tilde{F} \sin \frac{n\pi x}{X} + \sum_{\nu=1}^{\infty} \frac{1}{M_\nu} \hat{F} \phi_\nu$$

where

$$\hat{F} = \int_0^X F \phi_\nu \, dx$$

and M_ν is the norm of $\{\phi_\nu\}$, i.e.,

$$M_\nu = \int_0^X \phi_\nu^2 \, dx$$

The complementary transform $\widehat{(\ )}$ has the property that

$$\widehat{F''} = \phi'_\nu(0)\, F(0) - \phi'_\nu\,(X)\, F(X) - \xi_\nu^2 \, \hat{F}$$

Consider now the following simple boundary value problem:

$$\begin{cases} \dfrac{\partial F}{\partial t} = \dfrac{\partial^2 F}{\partial x^2} + G(x,t)\ , \quad 0 < x < X,\ t > 0 \\ x = 0 : \quad F = a(t) \\ x = X : \quad F = b(t) \\ t = 0 : \quad F = c(x) \end{cases}$$

Applying the transform $\widetilde{(\ )}$ to this system, we have

$$\begin{cases} \dfrac{d\tilde{F}}{dt} = \dfrac{n\pi}{X}\left[b(t) - a(t) \right] - \dfrac{n^2\pi^2}{X^2}\tilde{F} + \tilde{G}(n,t) \\ t = 0 : \quad \tilde{F} = \tilde{c}(n) \end{cases}$$

from which $\tilde{F}$ can be easily solved. Similarly, an application of $\widehat{(\ )}$ yields

$$\begin{cases} \dfrac{d\hat{F}}{dt} = \phi'_\nu(0)\, a(t) - \phi'_\nu\,(X)\, b(t) - \xi_\nu^2\,\hat{F} + \hat{G}(\,\xi_\nu,\ t) \\ t = 0 : \quad \hat{F} = \hat{c}(\,\xi_\nu) \end{cases}$$

from which $\hat{F}$ can also be solved. The solution F of the original problem is then

$$F(x,t) = \frac{2}{X} \sum_{n=1,3,5}^{\infty} \tilde{F} \sin \frac{n\pi x}{X} + \sum_{\nu=1}^{\infty} \frac{1}{M_\nu} \hat{F} \phi_\nu$$

with possible Gibbs' phenomenon on $x = 0$ and X.

But we have assumed that we do not know the set $\{\phi_\nu\}$. We wish, therefore, to establish the conditions under which the second series in the formula for F vanishes. This obviously requires that $\hat{F} = 0$. A sufficient condition for this to happen is

$$a(t),\ b(t),\ \hat{G}(\xi_\nu,t),\ \hat{c}(\xi_\nu) = 0$$

In other words, (1) the boundary values must be zero on $x = 0$ and X; (2) the inhomogeneous term and the initial value as functions of x must belong to a class which are expandable in terms of $\sin \frac{n\pi x}{X}$ with $n = 1,3,5, \ldots$. The most important member of this class is the constant

$$c = \frac{4c}{\pi} \sum_{n=1,3,5}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{X}$$

which yields

$$\begin{cases} \tilde{c} = \dfrac{2Xc}{\pi n}, & n = 1,3,5, \ldots \\ \hat{c} = 0 \end{cases}$$

Another obvious member is $\sin \frac{\pi x}{X}$ which yields

$$\begin{cases} \widetilde{\sin \frac{\pi x}{X}} = \dfrac{X}{2}, & n = 1 \\ \qquad\qquad = 0, & n \neq 1 \\ \widehat{\sin \frac{\pi x}{X}} = 0 \end{cases}$$

The above conclusion is easily generalizable: A general linear problem of order 2j in x and containing only even-order derivatives with respect to x is solvable by an application of the transform $\widetilde{(\ )}$ alone, if (1) $\partial^{2i}F/\partial x^{2i}$, $i = 0,1,2,..,(j-1)$, vanish on $x = 0$ and X, (2) the inhomogeneous term and initial values as functions of x belong to the same class stated before.

An Equitriangular Transform.

Consider the Helmholtz equation

$$\nabla^2 f + \lambda^2 f = 0$$

in a equilateral triangle R (Fig. 1) with the boundary condition

$$f = 0 \quad \text{on } B$$

Here ∇^2 is the Laplacian and B is the boundary of R. Sen[4] has discovered a set of eigenfunctions for this system

$$\psi_n = \sin \lambda_n p_1 + \sin \lambda_n p_2 + \sin \lambda_n p_3$$

with eigenvalues

$$\lambda_n = \frac{2n\pi}{\sqrt{3}a} , \ n = 1,2,3, \ldots .$$

(See Fig. 1 for nomenclature.) Based on Sen's set, an integral transform can be defined as

$$\tilde{F}(n) = \int_R F(x,y)\ \psi_n\ dx\, dy$$

[4]Sen, B., "Note on Some Two Dimensional Problems of Elasticity Connected with Plates Having Triangular Boundaries", Bulletin of the Calcutta Mathematical Society, 26 (1934), pp. 65-72.

But then[5],

$$\widetilde{\nabla^2 F} = - \int_B F \, (\nabla \psi_n) \cdot \vec{N} \, ds - \lambda^2_n \tilde{F}$$

where $\vec{N}$ is the outward unit normal vector of B and ds is line element along B. We see that the transform of $\nabla^2 F$ does not recognize the detailed variation of F on B, but only a certain average of it. This is an indication that the set is incomplete.

To make maximum use of Sen's set, we can supplement it by its complementary set $\{\phi_\nu\}$ with eigenvalues ζ_ν to make it complete. Unlike in the previous case, $\{\phi_\nu\}$ and ζ_ν are really unknown to us. However, we can follow exactly the same procedure as before to find out under what conditions are boundary value problems solvable by using the transform $\widetilde{(\;)}$ alone. It is not necessary to repeat the argument here. The conclusion is:

The linear boundary value problem containing only powers of the Laplacian (to the jth power) is solvable by an application of $\widetilde{(\;)}$ alone, if (1) $(\nabla^2)^i$, $i = 0,1,2, \ldots (j-1)$, vanish on B, (2) the inhomogeneous term and initial values belong to the class of functions which can be expanded in a series of $(\sin \lambda_n p_1 + \sin \lambda_n p_2 + \sin \lambda_n p_3)$.

It is very gratifying to see that the important constant function belongs to this class[6].

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[5] For a discussion of general integral transforms in an arbitrary region, see Kaplan, S. and Sonnemann, G., "A Generalization of the Finite Integral Transform Technique and Tables of Special Cases", Proc. 4th Midwestern Conference on Solid Mechanics, University of Texas, 1959, pp. 497-513.

[6] Sen, ibid.

In the appended table[7], we have summarized the important properties of the transform $\widetilde{(\ )}$ for easy reference.

The physical problems that can be solved by this transform are many. Some examples are transient viscous flow with or without suspended particles, transient heat conduction or natural convection, combined natural and forced convection, all inside an equitriangular duct or column; and transient deflections of simply supported equitriangular plates or membranes. The results will be reported elsewhere. The object of this note is to describe the fundamental format. It is hoped that other known sets of eigenfunctions in the literature can be employed in the same manner to their maximum capabilities.

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[7] Mr. Roy W. Miller of NASA Lewis Research Center kindly compiled this table.

TABLE

Description	Symbol	Form
Eigenfunctions (trilinear form)	ψ_n	$\sin \lambda_n p_1 + \sin \lambda_n p_2 + \sin \lambda_n p_3$
Eigenfunctions (Cartesian form)	ψ_n	$\sin \lambda_n\left(\frac{a}{\sqrt{3}} - \frac{\sqrt{3}Y}{2} + \frac{X}{2}\right) + \sin \lambda_n\left(\frac{a}{\sqrt{3}} + \frac{\sqrt{3}Y}{2} + \frac{X}{2}\right) + \sin \lambda_n\left(\frac{a}{\sqrt{3}} - X\right)$
Eigenvalues	λ_n	$\frac{2n\pi}{\sqrt{3}a}$ $n = 1,2,3, \ldots$
Norm	N_n	$\frac{3\sqrt{3}a^2}{2}$
Transform of F	$\tilde{F}$	$\int_R F \psi_n \, dx \, dy$
Transform of the Laplacian[8] of F	$\widetilde{\nabla^2 F}$	$-\lambda_n^2 \tilde{F}$
Transform of a constant C	$\tilde{C}$	$\frac{6a}{\lambda_n} C$

TABLE (CONTINUED)

Description	Symbol	Form
Inversion[9] of $\tilde{F}$	F	$\frac{2}{3\sqrt{3}a^2} \sum_{n=1}^{\infty} \tilde{F} \psi_n$
Integration of ψ_n over the equitriangular region	$\int_R \psi_n \, dx\, dy$	$\frac{6a}{\lambda_n}$
Transform of $p_1 p_2 p_3$	$\widetilde{p_1 p_2 p_3}$	$\frac{6\sqrt{3} a^2}{\lambda_n^3}$
Transform of $p_1 p_2 p_3 \left(p_1^2 + p_2^2 + p_3^2 - 3a^2\right)$	-	$\frac{144\sqrt{3} a^2}{\lambda_n^5}$

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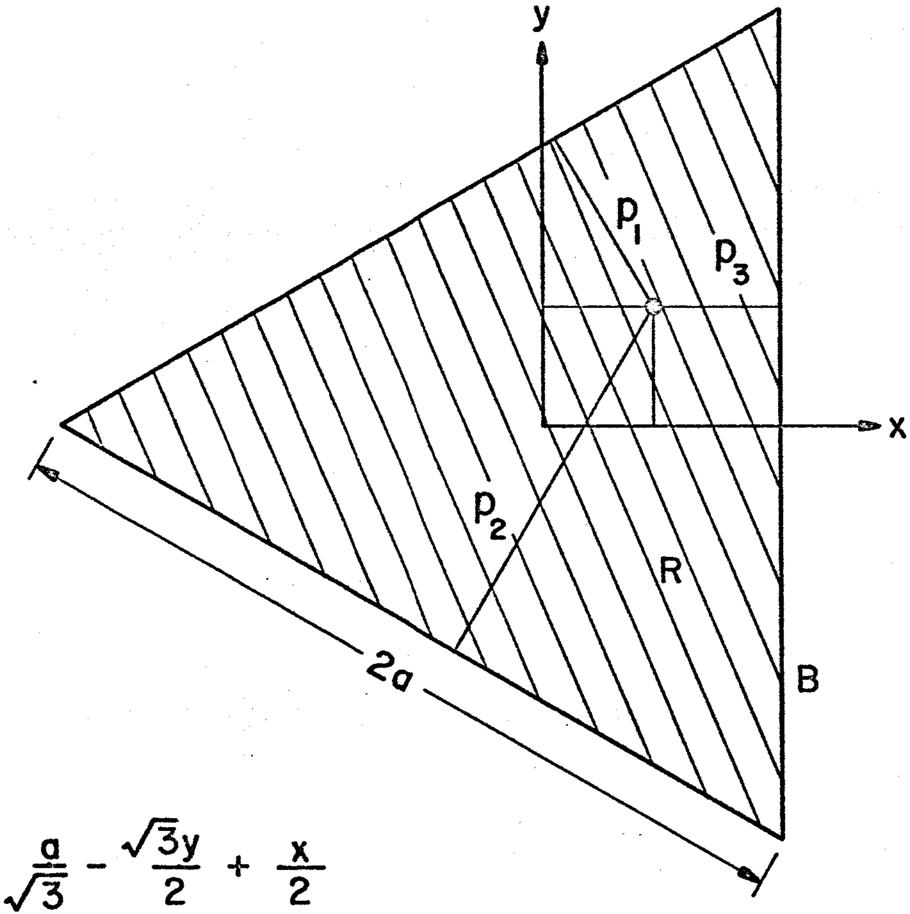
[8] F here must vanish on the sides of the triangle

[9] F here must be the solution of a boundary value problem satisfying the two requirements stated in the text.



$$p_1 = \frac{a}{\sqrt{3}} - \frac{\sqrt{3}y}{2} + \frac{x}{2}$$

$$p_2 = \frac{a}{\sqrt{3}} + \frac{\sqrt{3}y}{2} + \frac{x}{2}$$

$$p_3 = \frac{a}{\sqrt{3}} - x = \sqrt{3}\,a - (p_1 + p_2)$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$B\colon\ p_1 = 0,\ p_2 = 0,\ p_3 = 0$$

Fig. 1 The Equitriangular Region